The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
September 5, 2017

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1. Let G be a group of order 3^n, for an integer $n > 0$.

   (i) Prove that G has non-trivial centre Z.

   (ii) Prove that if G/Z is cyclic, then G is abelian.

   (iii) Prove that if G is not cyclic, it has at least 6 normal subgroups.

2. Let $\mathbb{Z}^2$ be the group of lattice points in the plane (ordered pairs of integers, with coordinatewise addition as the group operation). Let H_1 be the subgroup generated by the two elements $(2, 2)$ and $(2, -2)$, and H_2 the subgroup generated by the two elements $(2, 2)$ and $(1, 5)$.

   (i) Are the groups H_1 and H_2 isomorphic?

   (ii) Are the groups $G_1 = \mathbb{Z}^2/H_1$ and $G_2 = \mathbb{Z}^2/H_2$ isomorphic?

3. (i) Prove that $\sin 72°$ is an algebraic number, and find its minimal polynomial. Express $\sin 72°$ in terms of radicals.

   (ii) Find the Galois group of the splitting field of $\sin 72°$.

4. (10 points) Let $A \in \mathbb{R}^{3\times 3}$ be an orthogonal matrix (i.e., $A^T = A^{-1}$). Suppose that $\frac{1+i}{\sqrt{2}}$ is one eigenvalue of A.

   (i) What are eigenvalues of A^2?

   (ii) Is there necessarily a solution to the following equation?

   $$A^{38} = \alpha A^4 + \beta A^2 + \gamma I.$$

   If so, find $\alpha, \beta, \gamma \in \mathbb{R}$ that solve the equation.
   Hint: The Cayley-Hamilton theorem will help determine the answer.

5. (10 points) Let

   $$A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & 4 & 8 \\ 4 & 8 & 16 \end{bmatrix}.$$

   (i) What is $\mathrm{rank}(A)$?

   (ii) Find an orthonormal basis of eigenvectors for A.

   (iii) Let $B = \exp(A)$. What is the value of $B_{1,1}$ (the upper left entry of B)?

6. (10 points) Define the map $f : \mathbb{R}^{n\times n} \to \mathbb{R}^{n\times n}$ by the equation given in each of the following sub-problems. Determine whether the map is a linear transformation. If not, show that it is not. If so, determine the dimension of the image (also called the range) and the dimension of the kernel (also called the null space). Below $A \in \mathbb{R}^{n\times n}$.

   (i) $f(A) = A + A^{-1}$.

   (ii) $f(A) = Avv^T$ where $v \in \mathbb{R}^n$ and $v \neq 0$. Also, v^T is the transpose of v.

   (iii) $f(A) = D\cdot A - D^{-1}\cdot A^T$, where D is a diagonal matrix satisfying $D_{i,i} = (-1)^i$, $i = 1, 2, \ldots, n$.