UBC Department of Mathematics Qualifying Exam in Analysis

January, 2017

Every problem is worth 10 points.

Problem 1: Let $A = \begin{bmatrix} 2 & 2 \\ 2 & -1 \end{bmatrix}$, and denote $\vec{x}(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

1. Find the general solution of the system $\frac{d}{dt}\vec{x} = A\vec{x}$.

2. Solve the initial value problem

$$\frac{d}{dt}\vec{x} = A\vec{x} + \begin{bmatrix} t \\ 0 \end{bmatrix}, \quad \vec{x}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Problem 2: Consider the following ODE for $x(t)$: $\frac{d^2x}{dt^2} + \frac{dx}{dt} = x^3 - x$.

1. Rewrite this ODE as a system of two first-order equations.
2. Find all the critical points of the resulting system, and classify them as sinks, sources, saddles, or centres.
3. Show that if $x(t)$ is a solution, the quantity $\left(\frac{dx}{dt}\right)^2 + x^2 - \frac{1}{2}x^4$ is non-increasing.
4. Prove that the constant solution $x(t) \equiv 0$ is *stable* in this sense: for any $\epsilon > 0$, every solution for which $(x(0))^2 + \left(\frac{dx}{dt}(0)\right)^2$ is sufficiently small, satisfies $(x(t))^2 + \left(\frac{dx}{dt}(t)\right)^2 < \epsilon$ for all $t > 0$.

Problem 3: Consider the ODE eigenvalue problem:

$$(Ly)(x) = -(1+x^2)y''(x) + A(x)y'(x) = \lambda\, y(x), \quad 0 < x < 1,$$

with either Dirichlet BCs: $y(0) = y(1) = 0$; or Neumann BCs: $y'(0) = y'(1) = 0$.

1. Find the function $A(x)$ such that the ordinary differential operator L is self-adjoint (ie of Sturm-Liouville type) for either type of BC. For the rest of this question, take $A(x)$ to be that function.

2. Listing the eigenvalues λ in order: $\lambda_0 \leq \lambda_1 \leq \lambda_2 \leq \cdots$, show that for Dirichlet BCs $\lambda_0 > 0$, and for Neumann BCs $\lambda_1 > 0$.

3. Find the solution $u(x,t)$ of the following initial-boundary-value PDE problem in terms of the eigenvalues $\lambda_0, \lambda_1, \lambda_2, \ldots$ and corresponding eigenfunctions (but you need not actually *find* these eigenvalues and eigenfunctions):

$$\begin{aligned} &u_t = (1+x^2)u_{xx} - A(x)u_x, \quad 0 < x < 1, \;\; t > 0 \\ &u(0,t) = 1, \; u(1,t) = 2, \qquad u(x,0) = 1 + x. \end{aligned}$$

Problem 4: Find a matrix, $A \in \mathbb{R}^{2\times 2}$, satisfying

$$A = A^T, \quad A_{1,1} + A_{2,2} = 5, \quad \sum_{i,j} A_{i,j} = 19, \quad -A_{1,1} + A_{2,1} + A_{1,2} = 11.$$

Answer: This is a set of 4 linear equations with 4 unknowns. It can be solved via Gaussian elimination to give

$$A = \begin{bmatrix} 3 & 7 \\ 7 & 2 \end{bmatrix}.$$

Problem 5: Let $\mathcal{P}_2$ be the space of polynomials $a + bx + cx^2$ of degree at most 2 and with the inner product

$$\langle p, q \rangle = \int_{-1}^{1} p(x) \cdot q(x) dx.$$

1. Give an orthonormal basis for the orthogonal complement of $\text{span}(x)$.

2. Let l be the functional defined by $l(p) := p(0)$ for each $p \in \mathcal{P}_2$. Find $h \in \mathcal{P}_2$ so that $l(p) = \langle h, p \rangle$ for each $p \in \mathcal{P}_2$.

Problem 6: Let $A, B \in \mathbb{R}^{3\times 3}$. Let $I \in \mathbb{R}^{3\times 3}$ be the 3 by 3 identity matrix. Suppose that A has eigenvalues $\{-1, 4, 10\}$ and B has eigenvalues $\{-2, 4, 7\}$. For each of the following matrices, if possible determine the eigenvalues. If not, state that there is insufficient information to determine the eigenvalues.

(a) A^2.

(b) $A \cdot B$.

(c) $A + B$.

(d) $A - 5 \cdot I$.

(e) $A + A^{-1}$.