UBC Department of Mathematics Qualifying Exam in Analysis

September 6, 2016

Every problem is worth 10 points.

Problem 1: Let $\sum_{n=1}^{\infty} a_n$ be a convergent series of positive terms. Show that the series $\sum_{n=1}^{\infty} a_n^{\frac{n}{n+1}}$ also converges.

Hint: consider the set $S := \{n : a_n^{\frac{1}{n+1}} \leq 1/2\}$.

Problem 2: Find all entire functions $f(x+iy) = u(x) + iv(y)$, with the real valued functions $u(x)$ and $v(y)$ depending only on x and y respectively.

Problem 3: Consider the vector field $F = \langle xy^2+z, x^2y+2, x\rangle$. Evaluate the line integral $\int_C F \cdot dr$ where $C = (3^{t/\pi}, \sin t, \cos t)$ for $t \in [0, \pi]$.

Problem 4: Evaluate $I = \displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+1)(x^2+4)}\, dx$.

Problem 5: Let $f(x)$ be a uniform limit of real differentiable functions $f_n(x)$ on $[-1, 1]$. Assume that $|f_n'(x)| \leq C$ for some C independent of n and $x \in [-1, 1]$. Recall that, under these assumptions, $f(x)$ is always continuous.

(a) Is the function $f(x)$ necessarily differentiable? If so, prove it. If not, provide a counterexample.

(b) Suppose that, in addition, the derivatives $f_n'(x)$ converge pointwise to $g(x)$ and $f(x)$ is differentiable. Then, is it necessarily true that $f'(x) = g(x)$? If so, prove it. If not, provide a counterexample.

Problem 6: Find the maximum value of $\left|(1-z)e^z\right|$ in the region $|z| \leq 1$.