The University of British Columbia
Department of Mathematics
Qualifying Examination—Algebra
January 9, 2016

---

1. Let J denote the $m \times m$ matrix of 1's.
   (a) (6 points) Show that J is a diagonalizable matrix. Give a basis for $\mathbf{R}^m$ consisting of eigenvectors for J.
   (b) (4 points) Show that if we have an $m \times n$ matrix A with $AA^T = 2J + 5I$ then $n \geq m$. (Fischer's inequality)

2. (a) (5 points) Given three mutually orthogonal vectors in $\mathbf{R}^3$, we can determine the matrices representing orthogonal projection onto each. What is the sum of the three matrices?
   (b) (5 points) We say $(a_1, a_2, a_3, \ldots)$ is a *fibonacci sequence* of real numbers if it satisfies the fibonacci recurrence namely if $a_{i+2} = a_{i+1} + a_i$ for $i = 1, 2, 3, \ldots$. Let U be the set of fibonacci sequences. Show that U is a vector space over $\mathbf{R}$ where we can define the addition of two sequences in the obvious way. Give the dimension of U.

3. Let A be an $n \times n$ matrix with real entries. Suppose $A^2 = -I$.
   (a) (2 points) Show that A is invertible (or *nonsingular*).
   (b) (2 points) Show that A has no real eigenvalues.
   (c) (3 points) Show that n must be even.
   (d) (3 points) Show that $\det(A) = 1$.

4. (10 points) Calculate the addition and multiplication tables for the field $\mathbb{F}_4$.

5. (a) (7 points) Show that the group $SL_2(\mathbb{F}_4)$ naturally injects into the symmetric group S_5.
   (b) (3 points) Show that $SL_2(\mathbb{F}_4)$ is isomorphic to A_4 (the alternating group).

6. (10 points) Let R be a unitary commutative ring.
   (a) (2 points) Define the characteristic of R.
   (b) (3 points) Show that this characteristic of a field is either 0 or a prime number p.
   (c) (5 points) Under what conditions is $x \mapsto x^n$ $(n \in \mathbb{N})$ an endomorphism of R? Explain.