Qualifying Exam Problems: Linear Algebra and Differential Equations
(Jan 10, 2015)

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1. a) (5 points) Let $A \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix with all eigenvalues greater than or equal to 0. Show that there exists a square matrix B with $A = B^T B$.

   b) (3 points) show that for any square matrix $C \in M_{n\times n}(\mathbb{R})$, the matrix $C^T C$ is a symmetric matrix with all eigenvalues greater than or equal to 0.

   c) (2 points) Find the Jordan Canonical form of the matrix

$$A = \begin{pmatrix} 2 & 2 & 3 \\ 1 & 3 & 3 \\ -1 & -2 & -2 \end{pmatrix}$$

Solution: If A is a symmetric matrix then there exists and orthogonal matrix Q and a diagonal matrix D (of eigenvalues $\lambda_1 \geq 0$, $\lambda_2 \geq 0$, $\lambda_3 \geq 0, \ldots$) such that $AQ = QD$ or $A = QDQ^T$. Now since

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots \\ 0 & \lambda_2 & 0 & \cdots \\ 0 & 0 & \lambda_3 & \cdots \\ \vdots & \vdots & & \ddots \end{bmatrix} \text{ we may take } E = \begin{bmatrix} \sqrt{\lambda_1} & 0 & 0 & \cdots \\ 0 & \sqrt{\lambda_2} & 0 & \cdots \\ 0 & 0 & \sqrt{\lambda_3} & \cdots \\ \vdots & \vdots & & \ddots \end{bmatrix}$$

(using the fact that $\lambda_i \geq 0$) so that $EE^T = E^2 = D$. Then $A = QEE^TQ^T = B^TB$ for $B = E^TQ^T$.

Now if C^TC is symmetric since $\left(C^TC\right)^T = C^T\left(C^T\right)^T = C^TC$ and so C^TC has real eigenvalues. Let $(C^TC)\mathbf{x} = \lambda\mathbf{x}$ with $\mathbf{x} \neq \mathbf{0}$. But then $\mathbf{x}^T(C^TC)\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x}$ and so $||C\mathbf{x}|| = \mathbf{x}^T(C^TC)\mathbf{x} = \mathbf{x}^T\lambda\mathbf{x} = \lambda||\mathbf{x}||$. Given that $||C\mathbf{x}|| \geq 0$ and $||\mathbf{x}|| > 0$ we deduce that $\lambda \geq 0$.

For c), the characteristic polynomial is $\det(xI - A) = (x-1)^3$. One can then proceed to find the dim of the subspaces. Easy to check that the answer is

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

2. (10 points) Let

$$A = \begin{bmatrix} 0 & -2 & 1 \\ -2 & 3 & -2 \\ 1 & -2 & 0 \end{bmatrix}$$

The eigenvalues for A are -1,5. Determine an orthonormal basis for $\mathbb{R}^3$ that are eigenvectors for A and then give an orthogonal matrix Q and a diagonal matrix D so that $AQ = QD$.

Solution: By a variety techniques we can determine $\det(A - \lambda) = -(\lambda+1)^2(\lambda-5)$. Either use a straightforward calculation or recall that the determinant is the product of the eigenvalues or that the trace is the sum of the eigenvalues. The issue here is determining two orthogonal vectors in the eigenspace for $\lambda = -1$.

$$\lambda = -1 : \quad \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \quad \text{and } \lambda = 5 : \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

The students might use Gram Schmidt on the 2-dimensional eigenspace or perhaps using the cross product given two eigenvectors. Now the students need to remember to normalize to obtain

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \end{bmatrix}, \qquad D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

3. Let $A \in M_{n\times n}(\mathbb{R})$. Define the map $f : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ by

$$f(A) = A^T.$$

(a) (2 points) Show that f is linear.

Solution: We check that $f(A+B) = (A+B)^T = A^T + B^T = f(A) + f(B)$ and $f(kA) = (kA)^T = kA^T = kf(A)$.

(b) (3 points) Determine the dimension of the eigenspace of eigenvalue 1 for f.

Solution: If A is an eigenvector of eigenvalue 1 then $f(A) = A$ and so $A^T = A$ and so A is symmetric. The dimension of the space of symmetric matrices is $\binom{n}{2} + n$, namely the matrices $E_{ij} + E_{ji}$ for $i \neq j$ and E_{ii} (where E_{ij} is the matrix in $M_{n\times n}(\mathbb{R})$ with a 1 in position i, j and 0's elsewhere.

(c) (3 points) A matrix C is *skew symmetric* if $C^T = -C$. Determine the dimension of the eigenspace of eigenvalue -1 for f.

Solution: If A is an eigenvector of eigenvalue -1 then $f(A) = -A$ and so $A^T = -A$ and so A is skew symmetric. As above the dimension of the space of skew symmetric matrices is $\binom{n}{2}$, namely the matrices $E_{ij} - E_{ji}$ for $i \neq j$.

(d) (2 points) Show that any matrix $A \in M_{n\times n}(\mathbb{R})$ is a sum of a symmetric matrix B and a skew symmetric matrix C.

Solution: Using arguments about eigenspaces we note that the eigenspaces of different eigenvalues are linearly independent namely the eigenspaces for 1 and -1 generate a vector space of dimension $\binom{n}{2} + n + \binom{n}{2} = n^2$ which is the dimension of $M_{n\times n}(\mathbb{R})$. So a basis for the eigenspace for eigenvalue 1 and a basis for the eigenspace for eigenvalue -1 yield a basis for $M_{n\times n}(\mathbb{R})$ and so every $A \in M_{n\times n}(\mathbb{R})$ can be written as a sum $B + C$ where B is symmetric and C is skew symmetric.
Alternatively one can note that $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$.

4. (10 points) Show that $y_1(x) = x^2$ is one solution to the differential equation $x^2y'' - 3xy' + 4y = 0$. Use this as a starting point to find the general solution to the following second-order, nonhomogeneous differential equation with non-constant coefficients

$$x^2y'' - 3xy' + 4y = x^2 \ln x, (x > 0).$$

Solution:
Substitute $y_1 = x^2$ into the homogeneous equation, we obtain

$$LHS = x^2(2) - 3x(2x) + 4x^2 = 2x^2 - 6x^2 + 4x^2 = 0 = RHS.$$

To find the general solution to the nonhomogeneous equation, we need to find a second solution $y_2(x)$ to the homogeneous equation and one particular solution to the non homogeneous one.
Rewrite the equations in the following form

$$y'' - \frac{3}{x}y' + \frac{4}{x^2}y = y'' + p(x)y' + q(x)y = 0, \qquad \text{and} \qquad y'' - \frac{3}{x}y' + \frac{4}{x^2}y = \ln x.$$

Based on Abel's theorem, the Wronskian is

$$W[y_1, y_2] = y_1 y_2' - y_1' y_2 = e^{-\int p(x)dx} = e^{\int \frac{3}{x}dx} = x^3.$$

Thus,

$$x^2 y_2' - 2xy_2 = x^3. \quad \Rightarrow \quad y_2' - \frac{2}{x}y_2 = x.$$

Using method of integrating factors,

$$\left(x^{-2}y_2\right)' = \frac{1}{x} \quad \Rightarrow \quad y_2(x) = x^2 \ln x + Cx^2 = x^2 \ln x.$$

The second term is redundant to $y_1(x)$ thus is eliminated by choosing $C = 0$. Knowing two solutions of the homogeneous equation, one can find a particular solution to the nonhomogeneous equation using Variation of Parameters. That method yields

$$y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x),$$

where

$$u_1(x) = -\int \frac{y_2(x)g(x)}{W[y_1, y_2](x)}dx = -\int \frac{(\ln x)^2}{x}dx = -\frac{1}{3}(\ln x)^3.$$

$$u_2(x) = \int \frac{y_1(x)g(x)}{W[y_1, y_2](x)}dx = \int \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2.$$

Thus,

$$y_p(x) = -\frac{x^2}{3}(\ln x)^3 + \frac{x^2}{2}(\ln x)^3 = \frac{x^2}{6}(\ln x)^3.$$

Therefore,

$$y(x) = c_1 y_1(x) + c_2 y_2(x) + y_p(x) = c_1 x^2 + c_2 x^2 \ln x + \frac{x^2}{6}(\ln x)^3.$$

5. (10 points) Solve the wave equation

$$\begin{cases} \partial_{tt} u = 4\partial_{xx} u + xt, \quad 0 < x < \pi,\ t > 0, \\ u(0,t) = 0,\ u(\pi,t) = 1,\ t > 0, \\ u(x,0) = x,\ 0 < x < \pi, \\ (\partial_t u)(x,0) = 1,\ 0 < x < \pi. \end{cases}$$

Solution: 1) Set $w(x) = \frac{x}{\pi}$ and $v = u - w$. Then

$$\begin{cases} v_{tt} = 4v_{xx} + xt, \\ v(0,t) = v(\pi,t) = 0, \\ v(x,0) = (1 - \frac{1}{\pi})x, \\ (\partial_t v)(x,0) = 1. \end{cases}$$

2) By using separation of variables, we expand $v(x,t)$ as

$$v(x,t) = \sum_{n=1}^{\infty} T_n(t) \sin nx.$$

Then

$$T_n'' + 4n^2 T_n = h_n(t), \;\; T_n(0) = a_n, \, T_n'(0) = b_n,$$

where

$$\begin{aligned} (1 - \frac{1}{\pi})x &= \sum_{n=1}^{\infty} a_n \sin nx, \\ 1 &= \sum_{n=1}^{\infty} b_n \sin nx, \\ xt &= \sum_{n=1}^{\infty} h_n(t) \sin nx. \end{aligned}$$

Easy to check that

$$\begin{aligned} a_n &= (-1)^{n+1} \frac{2(\pi - 1)}{n\pi}, \\ b_n &= \frac{2}{n\pi}(1 + (-1)^{n+1}), \\ h_n(t) &= (-1)^{n+1} \frac{2}{n} t. \end{aligned}$$

Solving the equation for T_n then gives

$$T_n(t) = (-1)^{n+1} \frac{2(\pi - 1)}{n\pi} \cos(2nt) + (\frac{1}{n^2\pi}(1 + (-1)^{n+1}) - \frac{(-1)^{n+1}}{4n^4}) \sin(2nt) + \frac{(-1)^{n+1}}{2n^3} t.$$

(To find the above one just note that $T_n(t) = C_1 \cos(2nt) + C_2 \sin(2nt) + At$, with At being the particular solution)

Finally $u(x,t)$ is given by the series:

$$u(x,t) = \frac{x}{\pi} + \sum_{n=1}^{\infty} \Big((-1)^{n+1} \frac{2(\pi - 1)}{n\pi} \cos(2nt) + (\frac{1}{n^2\pi}(1 + (-1)^{n+1}) - \frac{(-1)^{n+1}}{4n^4}) \sin(2nt) + \frac{(-1)^{n+1}}{2n^3} t \Big) \sin(nx).$$

6. (a) (2 points) Turn the following nonlinear, second-order differential equation into a system of two

first-order differential equations.

$$x'' - x + x^3 = 0.$$

Solution: $\begin{cases} x' = y, \\ y' = x - x^3. \end{cases}$

(b) (2 points) Find all the steady states (fixed points) of the system obtained in (a).

Solution: $(0,\ 0)$, $(\pm 1,\ 0)$.

(c) (3 points) Classify the type of all the steady states found in (b) and determine their stability.

Solution: $(0,\ 0)$ is a saddle point, unstable. $(\pm 1,\ 0)$ are both centers, neutral stability.

(d) (3 points) Find a a function $V(x, y)$ that is conserved by this system.

Solution:

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{x - x^3}{y} \quad \Rightarrow \quad y^2 + C = x^2 - \frac{x^4}{2} \quad \Rightarrow \quad V(x, y) = x^2 - \frac{x^4}{2} - y^2 = C.$$