Qualifying Exam Problems: Analysis
(Jan 10, 2015)

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1. (10 points) For each value of the real constant $a > 0$, discuss the convergence of the series

$$\sum_{n=1}^{\infty} \frac{a^n}{(n!)^{\frac{1}{n}}}.$$

Solution: By using the obvious inequality $n! \leq n^n$, we get

$$\frac{a^n}{(n!)^{\frac{1}{n}}} \geq \frac{a^n}{n}.$$

Thus if $a \geq 1$, then the series diverges.
On the other hand, if $0 < a < 1$, then

$$\frac{a^n}{(n!)^{\frac{1}{n}}} \leq a^n$$

and the series converges by using comparison test.

2. Let $\vec{i}$, $\vec{j}$, $\vec{k}$ be the usual unit vectors in $\mathbb{R}^3$. Let $\vec{F}$ be the vector field

$$(x^2 + y)\vec{i} + (xy)\vec{j} + (xz + z^2)\vec{k}.$$

a) (3 points) Compute $\nabla \times \vec{F}$.

b) (7 points) Compute the integral of $\nabla \times \vec{F}$ over the surface $x^2 + y^2 + z^2 = 4$, $z \geq 0$.

Solution:

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ x^2 + y & xy & xz + z^2 \end{vmatrix} = (-z)\vec{j} + (y-1)\vec{k}.$$

Let $\Omega = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 4, \ z \geq 0\}$, $D = \{(x, y, 0) \in \mathbb{R}^3 | x^2 + y^2 \leq 4\}$. Note that Ω and D have the same boundary. By using Stokes' Theorem, we get

$$\begin{aligned} \int_{\Omega} \nabla \times \vec{F} \cdot d\vec{S} &= \int_{\partial\Omega} \vec{F} \cdot d\vec{l} \\ &= \int_{\partial D} \vec{F} \cdot d\vec{l} \\ &= \int_{D} (\nabla \times \vec{F}) \cdot d\vec{S} \\ &= \int_{D} ((-z)\vec{j} + (y-1)\vec{k}) \cdot \vec{k} dxdy = -4\pi. \end{aligned}$$

3. (10 points) Let $f\colon \mathbb{R} \to \mathbb{R}$ be a twice differentiable function such that $f \geq 0$ and $f'' \leq 0$ everywhere. Prove that f must be a constant.

Solution: Let $x_0 \in \mathbb{R}$. Enough to show $f'(x_0) = 0$. Now observe that for any t, we have

$$0 \leq f(x_0 + t) = f(x_0) + f'(x_0)t + \frac{f''(\xi)}{2}t^2 \leq f(x_0) + f'(x_0)t.$$

Since t is arbitrary, the result follows.

4. (10 points) Three sets of entire functions are described below. For each set, do two things:

   (i) Explain why there is a parametric representation of the form

   $$f(z) = c_0 + c_1 z + \ldots + c_N z^N, \qquad (c_0, c_1, \ldots, c_N) \in S,$$

   where $N \geq 0$ is an integer and S is a subset of $\mathbb{C}^{1+N}$.

   (ii) Describe the value of N and the conditions defining S as completely as possible.

   Here are the sets:

   (a) All entire functions f such that $\operatorname{Im}\{f(z)\} \leq 0$ for all $z \in \mathbb{C}$.

   (b) All entire functions f such that $|f(z)| \leq 2015 + |z|^{10}$ for all $z \in \mathbb{C}$.

   (c) All entire functions f such that $|f''(z)| \leq |z|$ for all $z \in \mathbb{C}$.

Solution:

(a) Given any such f, let $g(z) = \exp(-if(z))$. Then g is entire, with

$$|g(z)| = e^{\operatorname{Im}\{f(z)\}} \leq 1, \qquad z \in \mathbb{C}.$$

By Liouville's Theorem, g must be constant. Since f is continuous, it follows that f must also be constant. To match the requested pattern, take $N = 0$ and let S denote the set of $c \in \mathbb{C}$ where $\operatorname{Im}\{c\} \leq 0$.

(b) A direct application of the Extended Liouville Theorem implies that any f satisfying the given condition is a polynomial of degree at most 10. So $N = 10$ will work in the desired representation. A detailed description of S is not possible.

(c) Any f of the given family will make $g(z) = f''(z)/z$ analytic at all points $z \neq 0$, and bounded in a neighbourhood of $z = 0$. Therefore g has a removable singularity at 0 and we can treat g as if it were entire. With this interpretation,

$$|g(z)| \leq 1, \qquad z \in \mathbb{C},$$

so Liouville's Theorem implies that $g(z) = k$ for some complex k with $|k| \leq 1$. Consequently $f''(z) = kz$, which leads to

$$f'(z) = \frac{k}{2}z^2 + c_1, \qquad f(z) = \frac{k}{6}z^3 + c_1 z + c_0.$$

Thus $N = 3$ fits the desired pattern, with

$$S = \left\{(c_0, c_1, c_2, c_3) :\ c_2 = 0,\ |c_3| \leq \frac{1}{6}\right\}.$$

5. (a) (10 points) For each real constant a in the interval $-1 < a < 1$, present simple closed-form expressions for the integrals below:

$$I(a) = \int_0^{2\pi} \frac{d\theta}{1 + a\sin\theta}, \qquad J(a) = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}.$$

(b) Evaluate $I(4i/3)$, where I denotes the integral defined in part (a).
Note: Since the input $a = 4i/3$ does not obey the assumptions in part (a), a complete solution must *interpret* and *explain* the term "evaluate" as well as producing a numerical value.

Solution:

(a) One has $I(a) = J(a)$ for all a, thanks to the change of variable $\phi = \theta - \pi/2$. So focus on $I(a)$, recognizing $\sin\theta = \left(e^{i\theta} - e^{-i\theta}\right)/(2i)$. The parametrization $z = e^{i\theta}$ makes $dz = ie^{i\theta}\,d\theta$, so $d\theta = dz/(iz)$ and

$$I(a) = \int_{|z|=1} \frac{dz/(iz)}{(1 + a(z - 1/z)/(2i))} = \int_{|z|=1} \frac{2\,dz}{az^2 + 2iz - a} = \int_{|z|=1} f(z)\,dz,$$

where

$$f(z) := \frac{2}{az^2 + 2iz - a} = \frac{2/a}{(z - z_0)(z - z_1)}.$$

The poles of f can be determined using the quadratic formula:

$$z = \frac{-2i \pm \sqrt{-4 + 4a^2}}{2a} = \frac{i}{a}\left[-1 \pm \sqrt{1 - a^2}\right].$$

Both are purely imaginary; we name them $z_0 = \frac{i}{a}\left[-1 + \sqrt{1 - a^2}\right]$, $z_1 = \frac{i}{a}\left[-1 - \sqrt{1 - a^2}\right]$. Now

$$|z_1| = \frac{1 + \sqrt{1 - a^2}}{|a|} \geq \frac{1}{|a|} > 1,$$

so z_1 lies outside the disk of interest, and (from the factorization above)

$$|z_0\, z_1| = |-1| = 1 \quad \Longrightarrow \quad |z_0| = \frac{1}{|z_1|} < 1.$$

It follows that $I(a) = 2\pi i \operatorname{Res}(f; z_0)$. To find this residue, suppose A and B make

$$\frac{2/a}{(z - z_0)(z - z_1)} = f(z) = \frac{A}{z - z_0} + \frac{B}{z - z_1}.$$

Then $2/a = A(z - z_1) + B(z - z_0)$, and sending $z \to z_0$ gives

$$\operatorname{Res}(f; z_0) = A = \frac{2/a}{z_0 - z_1} = \frac{1}{i\sqrt{1 - a^2}}.$$

Finally, recalling $I(a) = 2\pi i \operatorname{Res}(f; z_0)$,

$$J(a) = I(a) = \frac{2\pi}{\sqrt{1 - a^2}}.$$

(b) Analytic extension of $I(z)$ from the real interval $-1 < z < 1$ to a superset having nonempty interior in $\mathbb{C}$ requires some kind of branch cut linking the points $z = \pm 1$. Go the long way, discarding all points $z = x + i0$ for which $|x| \geq 1$. (Sketch.) Then

$$I(4i/3) = \frac{2\pi}{\sqrt{1 + (16/9)}} = \frac{6\pi}{5}.$$

6. (10 points) Prove that this equation has precisely four solutions in the annulus $\frac{3}{2} < |z| < 2$:

$$z^5 + 15z + 1 = 0.$$

Include a statement of the main theorem (or theorems) on which your analysis is based.

Solution: This is a double application of Rouché's Theorem. A simple form of this result says, "*Let γ be a simple closed curve. Suppose f and g are analytic at all points on and inside γ, and*

$$|f(z) - g(z)| < |g(z)|, \qquad z \in \gamma.$$

Then f and g have the same number of zeros of f inside γ, counted according to multiplicity." (There is a more elaborate form, which allows a finite number of poles for f and g inside γ.)

We use $f(z) = z^5 + 15z + 1$ in both cases.

First, take $g(z) = 15z + 1$ and let γ be the circle where $|z| = 3/2$. Clearly g has exactly one zero inside γ, at $z = -1/15$. And on γ, the triangle inequality gives both

$$|g(z)| = |15z + 1| \geq 15\,|z| - 1 = 15\left(\frac{3}{2}\right) - 1 = \frac{43}{2} \geq \frac{42}{2} = 21$$

and

$$|f(z) - g(z)| = \left|z^5\right| = |z|^5 = \frac{3^5}{2^5} = \frac{243}{32} \leq \frac{256}{32} = 8.$$

Thus the conditions for Rouché's Theorem are in force, and we deduce that f has exactly one zero in the set where $|z| < 3/2$.

Second, take $g(z) = z^5 + 15z$ and let γ be the circle where $|z| = 2$. This time each z on γ obeys

$$|g(z)| = \left|z^5 + 15z\right| \geq |z|\left(|z|^4 - 15\right) = 2\,(16 - 15) = 2$$

and

$$|f(z) - g(z)| = 1.$$

Thus the conditions for Rouché's Theorem are in force, and we deduce that f has the same number of zeros as g has inside γ. Clearly $g(z) = z(z^4 + 15)$ has one zero at the origin and another four on the circle $|z| = 15^{1/4} < 2$, so f has 5 zeros with $|z| < 2$.

Combining the results above, we find that all 5 roots of f obey $|z| < 2$, and exactly one satisfies $|z| < 3/2$. So there are exactly 4 zeros obeying $3/2 \leq |z| < 2$. To get the chain of strict inequalities requested in the setup, it would suffice to re-run the first application of Rouché's Theorem on any circle of radius slightly larger than $3/2$. The gap between 21 and 8 noted above is positive, so there exists some $\epsilon > 0$ for which the desired inequality remains valid on $|z| = \frac{3}{2} + \epsilon$, and this completes the proof.